ON RESIDUAL PROPERTIES OF GENERALIZED HYDRA GROUPS

GILBERT BAUMSLAG AND ROMAN MIKHAILOV

Abstract. In this paper, the residual properties of groups defined by basic commutators as relations are studied. It is shown that the Hydra groups as well as certain of their generalizations and quotients are, in the main, residually torsion-free nilpotent. By way of contrast we give an example of a group defined by two basic commutators which is not residually torsion-free nilpotent.

1. Hydra groups

In a recent paper [4], Dyson and Riley introduced a family of one-relator groups

$$G_k = \langle a, b \mid [a, \underbrace{b, \dots, b}_{k}] = 1\rangle \ (k > 1)$$

which they called Hydra groups; here, as usual, if $x, y, a_1, \dots, a_k$ are elements of a group, $[x, y] = x^{-1}y^{-1}xy$, $x^y = y^{-1}xy$ and $[a_1, a_2, \dots, a_k] = [[[a_1, a_2], \dots, a_{k-1}], a_k]$.

The interest in these Hydra groups stems from the remarkable fact that, for each $k > 2$, their subgroups H_k generated by the elements $ba_1, \dots, ba_k$, where $a_i = [a, \underbrace{b, \dots, b}_{i-1}]$, have huge distortion. These Hydra groups are infinite cyclic extensions of finitely generated free groups and so are seemingly closely related to free groups, although their finitely generated subgroups have at most linear distortion. Here we will prove that these Hydra groups share a property of free groups:

Theorem 1.1. *For $k \geq 1$, the Hydra group*

$$\langle a, b \mid [a, \underbrace{b, \dots, b}_{k}] = 1\rangle$$

is residually torsion-free nilpotent, i.e., the intersection of their normal subgroups with torsion-free nilpotent quotients, is trivial.

This result is essentially contained in [2]. Since we will need to make use of a proof of Theorem 1.1, we will provide a such proof here.

2. Generalizations of Hydra groups

The Hydra groups can be viewed as special cases of one-relator groups defined by left-normed commutators of various weights in disjoint sets of variables. Our main result here is the proof of the following theorem, which is a considerable generalization of Theorem 1.1.

The research of the first author is supported by Grant CNS 111765.

Theorem 2.1. *Let X and Y be disjoint sets of variables and let u be an element in the free group on X and v and element in the free group on Y. If u and v are not proper powers, then*

$$\langle X, Y \mid [u, \underbrace{v, \ldots, v}_{k}] = 1 \rangle$$

is residually a finite p-group for every choice of the prime p.

The following theorem [3] is a special case of Theorem 2.1 which we will also need in the proof of Theorem 2.1, which because of its special nature, takes on a sharper form.

Theorem 2.2.
Let

$$G = \langle x_1, \ldots, x_n, t \mid [w(x_1, \ldots, x_n), t] = 1 \rangle$$

If w is not a proper power, then G is residually free, i.e., the intersection of the normal subgroups of G with free quotients is trivial.

There are a number of variations of Theorem 2.1 which involve certain quotients of the Hydra groups, which consist of groups defined by more than a single relation.

Here is a sample of such variations, followed by two rather surprising examples of some similarly defined groups.

- The groups
$$\langle a, b \mid [[a, b], [a, b, b, b]] = 1, [a, b, b, b, b] = 1 \rangle$$
are residually torsion-free nilpotent.
- The group
$$\langle a, b \mid [a, b, b] = [a, b, a, a, a] = 1 \rangle$$
is not residually torsion-free nilpotent.
- There exists a finitely-presented group with torsion whose defining relations are basic commutators.

The latter example is perhaps a little surprising. In particular this suggests a question we have yet to answer, namely whether a one-relator group defined by a basic commutator is residually finite or even residually nilpotent.

3. Hydra groups are residually torsion-free nilpotent

We prove here the afore-mentioned

Theorem 3.1. *The groups G_k are residually torsion-free nilpotent for all $k \geq 1$.*

We need the following theorem of P. Hall [7].

Theorem 3.2 (P. Hall)**.** *Let H be a group and $K \lhd H$ a normal nilpotent subgroup of H. If H/K and $H/[K, K]$ are nilpotent, then H is nilpotent.*

Proof. We construct a tower of torsion free nilpotent quotient groups H_i of G_k, $i \geq 2$, such that

$$\cap_{i \geq 2} \ker\{G_k \to H_i\} = \{1\}.$$

First observe that, by a standard argument using the Reidemeister-Schreier algorithm, we have the short exact sequence

$$1 \to F_k \to G_k \to \mathbb{Z} \to 1 \tag{3.1}$$

where F_k is the normal subgroup of G_k generated by $\{a, a^b, \ldots, a^{b^{k-1}}\}$. F_k is free on its given generators and G/F_k is infinite cyclic and generated by the image of b.

Let

$$H_i = G_k/\gamma_i(F_k). \tag{3.2}$$

We claim the groups H_i meet our criterion. Since $\cap_i \gamma_i(F_k) = \{1\}$, the intersection $\cap_{i\geq 2} \ker\{G_k \to H_i\}$ is trivial. Furthermore, the groups H_i are torsion free, since the groups $F_k/\gamma_i(F_k)$ are torsion-free (see e.g., [11]) It remains to show that each H_i is nilpotent. Consider the free generating set for $F_k/\gamma_i(F_k)$ given by the images of

$$c_1 = a,\ c_2 = [a, b],\ \ldots, c_j = [a, \underbrace{b, \ldots, b}_{j-1}],\ j = 1, \ldots, k.$$

The cyclic subgroup generated by the image of b acts on these generators as follows:

$$c_1 \mapsto c_1c_2 \tag{3.3}$$

$$\vdots \tag{3.4}$$

$$c_{k-1} \mapsto c_{k-1}c_k \tag{3.5}$$

$$c_k \mapsto c_k \tag{3.6}$$

The quotient group H_2 has the following presentation, where again we abuse notation and identify the various images of the elements of G_k with the elements themselves:

$$H_2 = \langle b, c_1, \ldots, c_k \mid [c_i, c_j] = [c_k, b] = 1,\ [c_i, b] = c_{i+1},\ i < k\rangle.$$

H_2 is nilpotent of class k. Applying Hall's Theorem 3.2, with

$$H = H_i \qquad \text{and} \qquad K = F_k/\gamma_i(F_k)$$

we conclude that

$$H_i \cong F_k/\gamma_i(F_k) \rtimes \mathbb{Z}$$

is nilpotent, proving the theorem. □

4. A residual property of some generalizations of the Hydra group

In this section we will prove the following residual property of a generalization of the Hydra groups.

Theorem 4.1. *2.1 Let $F(X)$ be the free group on X and let $F(Y)$ be the free group on Y. Furthermore let $u \in F(X)$, $v \in F(Y)$ and suppose that neither element is a proper power. Then the group*

$$V_k(u, v) := \langle X, Y \mid [u, \underbrace{v, \ldots, v}_{k}] = 1\rangle \tag{4.1}$$

is residually a finite p-group for every prime p whenever $k \geq 2$.

The proof of Theorem 4.1 depends on the notion of a p-preimage closed subgroup, a related theorem and a number of lemmas involving centralizers.

For a group G, a subgroup $H < G$ is called *p-preimage closed in G* if

$$H = \{g \in G \mid \text{for every } f : G \to \text{ finite } p-\text{group},\ f(g) \in f(H)\}$$

We will need the following results from [10] :

Theorem 4.2. *1) (A particular case of Theorem 4.2 in [10]) Let p be a prime, A, B be residually finite p-groups and $c \in A, d \in B$ are elements of infinite order, such that the subgroup $\langle c\rangle$ is p-preimage closed in A, $\langle d\rangle$ is p-preimage closed in B. Then the amalgamated product $A *_{c=d} B$ is residually finite p-group.*
2) (A particular case of Theorem 3.4 in [10]) Let p be a prime and G a residually finite p-group. The following assertions about the subgroup H of G are equivalent: (i) if G' is a copy of G and H' denotes the subgroup H in G', then $G *_{H=H'} G'$ is residually a finite p-group; (ii) subgroup H is p-preimage closed in G.

Next we need several lemmas about centralizers. To begin with we will need to prove that the centralizer of the element a in the Hydra group G_k is generated by a.

Lemma 4.1. *If $k > 1$, then the centralizer of $a \in G_k$ is generated by a.*

Proof. Suppose that $[g, a] = 1$ in G_k. Since $G_k = F_k \rtimes \langle t\rangle$, $g = hb^l$, $h \in F_k, l \in \mathbb{Z}$. Adopting the notation used in the proof of Theorem 3.1 we find that, modulo $\gamma_2(F_k)$,

$$g^{-1}ag = b^{-\ell}ab^{\ell} = ac_2^{\ell}d$$

where here d is a product of the images of the elements $c_3, \ldots$. Consequently, $\ell = 0$ and so $g \in F_k$. Since F_k is free, g is a power of a. □

One of the consequences of Lemma 4.1 is the next

Lemma 4.2. *Let $k \geq 2$, $v \in F(Y)$ an element which is not a proper power and let*

$$J = \langle Y, a, b \mid [a, \underbrace{b, \ldots, b}_{k}] = 1, b = v\rangle.$$

Then the centralizer of a in J is the cyclic subgroup $\langle a\rangle$.

Proof. Observe that J is an amalgamated product of G_k and $F(Y)$:

$$J = G_k *_{b=v} F(Y).$$

Suppose that g lies in the centralizer of a in J. Then g can be written in the form

$$g = e_1 \ldots e_n,$$

where each e_i is from one of the factors $F(Y)$ or G_k, and successive e_i, e_{i+1} come from different factors. Since $ag = ga$ and a does not belong to the subgroup generated by b it is not hard to see that $n = 1$. In this case, we have $ae_1 = e_1a$. It follows that $e_1 \in G_k$ in which case Lemma 4.1 applies which means that e_1 is a power of a as claimed. □

The next step in the proof of Theorem 2.1 makes use of Theorem 4.2.

Lemma 4.3. *The subgroup $\langle b\rangle$ is p-preimage closed in G_k.*

Proof. Finitely generated, residually torsion-free nilpotent groups are residually finite p-groups for every choice of the prime p [6]. So G_k is residually a finite p-group by Theorem 3.1. So by Theorem 4.2 2 $\langle b\rangle$ is p-preimage closed in G_k if and only if the group

$$\tilde{G}_k := \langle a, b, c \mid [a, \underbrace{b, \ldots, b}_{k}] = [c, \underbrace{b, \ldots, b}_{k}] = 1\rangle$$

is residually a finite p-group. In order to prove this, we follow the proof of Theorem 3.1. Now $\tilde{G}_k$ is the middle of a short exact sequence

$$1 \to F_{2k} \to \tilde{G}_k \to \mathbb{Z} \to 1$$

where the free group F_{2k} has generators $\{a, a^b, \ldots, a^{b^{k-1}}, c, c^b, \ldots, c^{b^{k-1}}\}$, and the cyclic quotient of $\tilde{G}$ by F_{2k} is generated by the image of b. It follows along the same lines as in the proof of Theorem 3.1, that the group $\tilde{G}_k$ is residually $F_{2k}/\gamma_i(F_{2k}) \rtimes \mathbb{Z}$-group for $i \geq 1$, and that the groups $F_{2k}/\gamma_i(F_{2k}) \rtimes \mathbb{Z}$ are torsion-free nilpotent for all $i \geq 1$. Hence $\tilde{G}_k$ is residually a finite p-group and therefore $\langle b\rangle$ is p-preimage closed in G_k. □

Next we prove the following

Lemma 4.4. *Let*

$$J = \langle Y, a \mid [a, \underbrace{v, \ldots, v}_{k}] = 1\rangle,$$

where v is not a proper power in $F(Y)$. Then J is residually a finite p-group for every prime p.

Proof. Observe that J is an amalgamated product:

$$J = F(Y) *_{v=b} G_k.$$

In order to prove that J is residually a finite p-group we again have to appeal to Theorem 4.2 (1). We have already proved that $\langle b\rangle$ is p-preimage closed in G_k. Now by Theorem 2.2, since v is not a proper power, $L = \langle Y, t \mid [t, v] = 1\rangle$ is residually free. So the subgroup M of L generated by Y and $t^{-1}Yt$ is residually a finite p-group. But M is an amalgamated product of two copies of $gp(Y)$ amalgamating $gp(v)$. Hence $gp(v)$ is p-preimage closed in $gp(Y)$. It follows that J is residually a finite p-group by Theorem 4.2 (2). □

The last step before we come to the proof of theorem 4.1 is

Lemma 4.5. *The subgroup of J generated by a is p-preimage closed in J.*

Proof. Suppose that $c \in J$ is such that for every homomorphism f from J into a finite p-group $f(c) \in f(\langle a\rangle)$. It follows that $[c, a]$ lies in every normal subgroup of J of index a power of p. Since J is residually a finite p-group, it follows that $[c, a] = 1$. So c is a power of a since the centralizer of a in J is generated by a. □

We are now in position to complete the proof of Theorem 4.1. Present $V_k(u, v)$ as an amalgamated product

$$V_k(u, v) = J *_{a=u} F(X).$$

Since J is residually a finite p-group and $\langle a\rangle$ is p-preimage closed in J, and $\langle u\rangle$ is p-preimage closed in $F(X)$, the group $V_k(u, v)$ is a residually finite p-group by Theorem 4.2 (1).

5. Some quotients of Hydra groups

We consider next various quotients of the Hydra groups which take the form

$$Q(r(a, b)) := \langle a, b \mid r(a, b) = 1,\ [a, \underbrace{b, \ldots, b}_{k}] = 1\rangle$$

where the $r(a, b) \in F(a, b)$ are suitably chosen.

Proposition 5.1. *Let $L = G_k/R$, for some $R \lhd G_k$ let A be a normal subgroup of L generated by aR . If L/A is infinite cyclic, generated by the image of the element b and if A is residually torsion-free nilpotent, then L is residually torsion-free nilpotent.*

Proof. Recall the notation which we used in (3.1):

$$G_k/F_k \simeq \langle b \rangle$$

So $A = F_kR/R$. Since A is a residually torsion-free nilpotent quotient of F_k and b acts nilpotently on the quotients of F_k on the lower central quotients of F_k, there exists a sequence of normal subgroups $A \rhd C_i \rhd C_{i+1} \rhd \dots$ such that $\cap_i C_i = \{1\}$ such that A/C_i is torsion-free nilpotent for all $i \geq 1$ and such that the image of b acts nilpotently on these quotients. This suffices then to ensure that L is residually torsion-free nilpotent. $\square$

Proposition 5.1 then allows us to construct some of the examples that we discussed above.

Example. Consider the groups

$$\langle a, b \mid [[a, \underbrace{b, \dots, b}_{k-2}], \underbrace{e, \dots, e}_{l}] = 1, [a, \underbrace{b, \dots, b}_{k}] = 1 \rangle$$

where $e = [a, \underbrace{b, \dots, b}_{k-1}]$ are residually torsion-free nilpotent for all $l \geq 1$. The simplest non-nilpotent group of such type is the following:

$$\langle a, b \mid [[a, b], [a, b, b]] = 1, [a, b, b, b] = 1 \rangle$$

To show that the above groups satisfy the hypothesis of Proposition 5.1, consider the generators of the subgroup F_k, given in the proof of Lemma 3.1:

$$c_1 = a,\ c_2 = [a, b],\ \dots, c_j = [a, \underbrace{b, \dots, b}_{j-1}],\ j = 1, \dots, k.$$

The action of $\langle b \rangle$ on these generators is given by (3.3)-(3.6). Writing the relator $[[a, \underbrace{b, \dots, b}_{k-2}], \underbrace{e, \dots, e}_{l}]$ in terms of the generators $c_1, \dots, c_k$, we find that $[c_{k-1}, \underbrace{c_k, \dots, c_k}_{l}]$ and its normal closure in G_k is cyclic. Recall that a free product of residually torsion-free nilpotent groups is residually torsion-free nilpotent. The group $A = F_kR/R$ (using the notation in Proposition 5.1) is the free product

$$F(c_1, \dots, c_{k-2}) * \langle c_{k-1}, c_k \mid [c_{k-1}, \underbrace{c_k, \dots, c_k}_{l}] = 1 \rangle$$

which is residually torsion-free nilpotent by Lemma 3.1.

Next we have

Example. For $s \geq 1$, consider the group

$$\langle a, b \mid [[a, \underbrace{b, \dots, b}_{s}], [a, \underbrace{b, \dots, b}_{k-1}] = 1, [a, \underbrace{b, \dots, b}_{k}] = 1 \rangle \tag{5.1}$$

Again, denoting the images of the c_i-s as before simply as c_i, we see that the subgroup $A = F_kR/R$ can be presented in the form

$$\langle c_1, \dots, c_k \mid [c_i, c_k] = 1,\ i = l, \dots, k-1 \rangle$$

which is isomorphic to the group $F(c_1, \dots, _{l-1}) * (F(c_l, \dots, c_{k-1}) \times \langle c_k \rangle)$, which is clearly residually torsion-free nilpotent. Conditions of the Proposition 3.1 are satisfied, hence the

group (5.1) is residually torsion-free nilpotent. A simple example of a group of this kind is

$$\langle a, b \mid [[a, b], [a, b, b, b]] = 1, [a, b, b, b, b] = 1\rangle.$$

Remark. Observe that, for $k \geq 1$, the groups

$$\langle a, b \mid [a, \underbrace{b, \ldots, b}_{k-1}, a] = 1, [a, \underbrace{b, \ldots, b}_{k}] = 1\rangle$$

are residually nilpotent by the following result from [12]: any central extension of a one-relator residually nilpotent group is residually nilpotent.

6. An example

As we noted at the outset, we have been unable to determine whether a one-relator group defined by a basic commutator is residually torsion-free nilpotent, or residually a finite p-group or even residually finite. The best that have managed to find is an example of a group defined by two relations, one of which is a basic commutator, which is not even residually a finite p-group.

Theorem 6.1. *The group*

$$G = \langle a, b \mid [a, b, b] = [a, b, a, a, a] = 1\rangle$$

is not residually torsion-free nilpotent. Moreover, it is not residually a finite p-group if $p \neq 2$.

Proof. We claim that if $w = [a, b, a, a, b, a]$, then

$$w \notin \gamma_7(G),\ \ w^2 \in \gamma_7(G).$$

This can be proved directly or by appealing to GAP, as the following GAP fragment shows:

```
gap> F:=FreeGroup(2);;
a:=F.1;; b:=F.2;;
gap> G:=F/[ LeftNormedComm([a,b,b]), LeftNormedComm([a,b,a,a,a])];;
gap> phi:=NqEpimorphismNilpotentQuotient(G,6);;
gap> aa:=Image(phi,G.1);;
bb:=Image(phi,G.2);;
gap> xx:=LeftNormedComm([aa,bb,aa,aa,bb,aa]);;
gap> Order(xx);
2
```

Now we will show that w is a generalized 2-torsion element. That is, for every $n \geq 1$, the order of w is a power of 2 modulo $\gamma_n(G)$.

The Hall-Witt identity together with the second relation implies that

$$[a, b, a, a, b, a]^{b^{-1}}[b, a^{-1}, [a, b, a, a]]^a = 1$$

Hence,

$$[a, b, a, a, b, a]^{b^{-1}}[[a, b], [a, b, a, a]] = 1 \tag{6.1}$$

It follows from the the Hall-Witt identity that

$$[a,b,a,a,[a,b]]^{a^{-1}}[a^{-1},[b,a],[a,b,a]]^{[a,b]}[a,b,[a,b,a]^{-1},a]^{[a,b,a]}=1. \tag{6.2}$$

and

$$[a,b,[a,[a,b]]]^{b^{-1}}[b^{-1},[a,b,a],a]^{[a,b,a]^{-1}}[a,[a,b],a^{-1},b^{-1}]^{a}=1,$$

Hence

$$\begin{aligned}[a,[a,b],a^{-1},b^{-1}]^{a}=&[[a,b,a,a^{-1}]^{[a,[a,b]]},b^{-1}]^{a}=\\ &[[a,b,a,a]^{-[a,[a,b]]},b^{-1}]^{a}=[[a,b,a,a]^{-[a,[a,b]]},b]^{-b^{-1}a}=\\ &[[a,b,a,a],b^{[a,b,a]}]^{[a,b,a,a]^{-1}[a,[a,b]]b^{-1}a}=\\ &[[a,b,a,a],[a,b,a]]^{d}[a,b,a,a,b]^{[a,b,a]d},\end{aligned}$$

where $d=[a,b,a,a]^{-1}[a,[a,b]]b^{-1}a$. The Hall-Witt identity together with (6.1), imply that

$$[[a,b,a,a],[a,b,a]]\in[\langle w\rangle^{G},G] \tag{6.3}$$

and therefore

$$[a,[a,b],a^{-1},b^{-1}]^{a}\equiv[a,b,a,a,b]^{[a,b,a]d}\mod[\langle w\rangle^{G},G]$$

We also have

$$[a,b,a,b^{-1}]^{a^{-1}}[a^{-1},b,[a,b]]^{b^{-1}}=1.$$

Hence

$$\begin{aligned}[b^{-1},[a,b,a]]=[b,a,a^{-1},[a,b]]^{b^{-1}a}=[[a,b,a]^{a},[a,b]]^{[b,a]b^{-1}a}&\\ \equiv[a,b,a,[a,b]]^{[b,a]b^{-1}a}&\mod\langle w\rangle^{G}.\end{aligned}$$

and therefore,

$$[b^{-1},[a,b,a],a]^{[a,b,a]^{-1}}\equiv[a,b,a,[a,b],a^{b[a,b]}]^{[b,a]b^{-1}a[a,b,a]^{-1}}\mod[\langle w\rangle^{G},G]$$

We conclude that

$$[a,b,[a,[a,b]]]\equiv[a,b,a,a,b]^{-[a,b,a]db}[a,b,a,[a,b],a^{b[a,b]}]^{[b,a]b^{-1}a[a,b,a]^{-1}b}\mod[\langle w\rangle^{G},G]$$

Hence

$$\begin{aligned}[a,b,[a,[a,b]]],a]&\equiv[[a,b,a,a,b]^{-[a,b,a]db}[a,b,a,[a,b],a^{b[a,b]}]^{[b,a]b^{-1}a[a,b,a]^{-1}b},a]\\ &\equiv w[[a,b,a,[a,b],a^{b[a,b]}]^{[b,a]b^{-1}a[a,b,a]^{-1}b},a]\mod[\langle w\rangle^{G},G]\end{aligned} \tag{6.4}$$

Denoting

$$h:=[a,b,a,[a,b]],$$

we can reformulate the above equivalence as

$$[a,b,[a,[a,b]]],a]\equiv w\mod[\langle w\rangle^{G},G][\langle h\rangle^{G},G,G].$$

Apply the Witt-Hall identity one more time (here we use the relation $[a,b,b]=1$):

$$[a,b,a,[a,b],b]^{[b,a]}[b,[a,b,a]^{-1},[a,b]]^{[a,b,a]}=1.$$

The equivalence (6.4) implies that

$$[a,b,a,[a,b],b]\in[\langle w\rangle^{G},G][\langle h\rangle,G,G]$$

That is, we have equivalences

$$[h, a] \equiv w \mod [\langle w\rangle^G, G][\langle h\rangle, G, G]$$
$$[h, b] \equiv 1 \mod [\langle w\rangle^G, G][\langle h\rangle, G, G]$$

This implies that, for every n,

$$[a, b, [a, [a, b]]], a] \equiv w \mod [\langle w\rangle^G, G]\gamma_n(G).$$

The second term of the relation (6.2) lies in $[\langle w\rangle^G, G]$ by (6.3). Now the relations (6.1) and (6.2) imply that, for every n,

$$w^2 \equiv [\langle w\rangle^G, G]\gamma_n(G).$$

□

Remark. Recall that, for a free group on two generators, the seventh term of the lower central series is the normal closure of all basic commutators of weight 7,8,9,10 (see [5]). It follows from the proof of Theorem 6.1 that the group

$$\langle a, b \mid [a, b, b] = [a, b, a, a, a] = 1, \text{ all basic commutators of weight 7,8,9,10}\rangle$$

has 2-torsion, namely

$$[a, b, a, a, b, a]^2 = 1, \ [a, b, a, a, b, a] \neq 1.$$

6.1. **Acknowledgements.** The second author thanks A. Talambutsa for helping with GAP.

References

[1] G. Baumslag: On the Residual Nilpotence of Certain One-Relator Groups, *Comm. Pure Appl. Math.* **21** (1968), 491-506.
[2] G. Baumslag: Recognizing powers in nilpotent groups and nilpotent images of free groups, *J. Aust. Math. Soc.* **83** (2): (2007), 149-155.
[3] G. Baumslag: On generalized free products, *Math. Zeitschr.* **78**, (1962), 423-438.
[4] W. Dison and T. Riley, Hydra groups, arxiv: 1002.1945, to appear in Comm. Math. Helv.
[5] D. Jackson, A. Gaglione and D. Spellman: Basic commutators as relators,*J. Group theory* **5**, (2002), 351-363.
[6] K. W. Gruenberg: Residual properties if infinite solvable groups, *Proc. London Math. Soc.* (3) **7** (1957), 29-62.
[7] P. Hall: Some sufficient conditions for a group to be nilpotent, *Illinois J. Math.* **2** (1958), 787–801.
[8] G. Higman: Amalgams of p-groups, *J. Algebra* **1** (1964), 301–305.
[9] P. Hilton: Nilpotent actions on nilpotent groups, *Lecture Notes in Mathematics*, **450** (1975), 174–196.
[10] G. Kim and J. McCarron: On amalgamated free-products of residually p-finite groups, *J. Algebra*, **162** (1993), 1–11.
[11] W. Magnus, A. Karrass, D. Solitar: Combinatorial group theory. Presentations of groups in terms of generators and relations, Reprint of the 1976 second edition, Dover Publications, Inc., Mineola, NY, 2004.
[12] R. Mikhailov: Residual nilpotence and residual solvability of groups, *Sb. Math.* **196** (2005), 1659–1675.

Dept. of Computer Science, City College of New York, Convent Avenue and 138th Street, New York, NY 10031
E-mail address: gilbert.baumslag@gmail.com

Chebyshev Laboratory, St. Petersburg State University, 14th Line, 29b, Saint Petersburg, 199178 Russia and St. Petersburg Department of Steklov Mathematical Institute
E-mail address: rmikhailov@mail.ru
URL: http://www.mi.ras.ru/~romanvm/pub.html